

Effects of Imperfect Gate Operations in Shor's Prime Factorization Algorithm

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(Dated: 2001)

The effects of imperfect gate operations in implementation of Shor's prime factorization algorithm are investigated. The gate imperfections may be classified into three categories: the systematic error, the random error, and the one with combined errors. It is found that Shor's algorithm is robust against the systematic errors but is vulnerable to the random errors. Error threshold is given to the algorithm for a given number N to be factorized.

PACS numbers: PACS numbers: 03.67.Lx, 89.70.+c, 89.80.+h

I. INTRODUCTION

Shor's factorization algorithm [1] is a very important quantum algorithm, through which one has demonstrated the power of quantum computers. It has greatly promoted the worldwide research in quantum computing over the past few years. In practice, however, quantum systems are subject to influence of environment, and in addition, quantum gate operations are often imperfect [2, 3]. Environment influence on the system can cause decoherence of quantum states, and gate imperfection leads to errors in quantum computing. Thanks to Shor's another important work, in which he showed that quantum error correlation can be corrected [4]. With quantum error correction scheme, errors arising from both decoherence and imperfection can be corrected.

There have been several works on the effects of decoherence on Shor's algorithm. Sun *et al.* discussed the effect of decoherence on the algorithm by modeling the environment [5]. Palma studied the effects of both decoherence and gate imperfection in ion trap quantum computers [6]. There have also been many other studies on the quantum algorithm [7, 8, 9, 10].

The error correction scheme uses available resources. Thus it is important to study the robustness of the algorithm itself so that one can strike a balance between the amount of quantum error correction and the amount of qubits available. In this paper, we investigate the effects of gate imperfection on the efficiency of Shor's factorization algorithm. The results may guide us in practice to suppress deliberately those errors that influence the algorithm most sensitively. For those errors that do not affect the algorithm very much, we may ignore them as a good approximation. In addition, study of the robustness of algorithm to errors is important where one can not apply the quantum error correction at all, for instance, in cases that there are not enough qubits available.

The paper is organized as follows. Section II is devoted to an outline of Shor's algorithm and different error's modes. In Section III, we present the results. Finally, a short summary is given in Section IV.

II. SHOR'S ALGORITHM AND ERROR'S MODES

Shor's algorithm consists of the following steps:

1) preparing a superposition of evenly distributed states

$$|\psi\rangle = \frac{1}{\sqrt{q}} \sum_{a=0}^{q-1} |a\rangle|0\rangle,$$

where $q = 2^L$ and $N^2 \leq q \leq 2N^2$ with N being the number to be factorized;

2) implementing $y^a \bmod N$ and putting the results into the 2nd register

$$|\psi_1\rangle = \frac{1}{\sqrt{q}} \sum_{a=0}^{q-1} |a\rangle|y^a \bmod N\rangle;$$

3) making a measurement on the 2nd register; The state of the register is then

$$|\phi_2\rangle = \frac{1}{\sqrt{A+1}} \sum_{j=0}^A |jr+l\rangle|z=y^l=y^{jr+l} \bmod N\rangle$$

where $j \leq \left[\frac{q-l}{r}\right] = A$.

4) performing discrete Fourier transformation (DFT) on the first register $|\phi_3\rangle = \left(\sum_c \tilde{f}(c)|c\rangle\right)|z\rangle$, where

$$\tilde{f}(c) = \frac{\sqrt{r}}{q} \sum_{j=0}^{\frac{q}{r}-1} \exp\left(\frac{2\pi i(jr+l)}{q}\right) = \frac{\sqrt{r}}{q} e^{\frac{2\pi i lc}{q}} \sum_{j=0}^{\frac{q}{r}-1} \exp\left(\frac{2\pi i jrc}{q}\right).$$

This term is nonzero only when $c = k\frac{q}{r}$, with $k = 0, 1, 2, \dots, r-1$, which correspond to the peaks of the distribution in the measured results, and thus this term becomes $\tilde{f}(c) = \frac{1}{\sqrt{r}} e^{\frac{2\pi i l c}{q}}$. The Fourier transformation is important because it makes the state in the first register the same for all possible values in the 2nd register. The DFT is constructed by two basic gate operations: the single bit gate operation $A_j = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, which is also called the Walsh-Hadamard transformation, and the 2-bits controlled rotation

$$B_{jk} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\theta_{jk}} \end{pmatrix}$$

with $\theta_{jk} = \frac{\pi}{2^{k-j}}$. The gate sequence for implementing DFT is

$$(A_{q-1})(B_{q-2q-1}A_{q-2}) \dots (B_{0q-1}B_{0q-2} \dots B_{01}A_0).$$

Errors can occur in both A_j and B_{jk} . A_j is actually a rotation about y-axis through $\frac{\pi}{2}$

$$A_j(\theta) = e^{\frac{i}{\hbar} S_y \theta} = I \cos\left(\frac{\theta}{2}\right) - i \sin\left(\frac{\theta}{2}\right) \sigma_y = \begin{pmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}$$

If the gate operation is not perfect, the rotation is not exactly $\frac{\pi}{2}$. In this case, A_j is a rotation of $\frac{\pi}{2} + 2\delta$

$$A_j(\delta) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos(\delta) - \sin(\delta) & -(\sin(\delta) + \cos(\delta)) \\ \sin(\delta) + \cos(\delta) & \cos(\delta) - \sin(\delta) \end{pmatrix}.$$

If δ is very small, we have:

$$A_j(\theta) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 - \delta & -(1 + \delta) \\ 1 + \delta & 1 - \delta \end{pmatrix}.$$

Similarly, errors in B_{jk} can be written as

$$B_{jk} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i(\theta_{jk} + \delta)} \end{pmatrix}.$$

With these errors, the DFT becomes

$$|a\rangle \rightarrow \frac{1}{\sqrt{q}} \sum_{c=0}^{q-1} e^{i(\frac{2\pi}{q} + \delta_c)a} (1 + \delta'_c) |\tilde{c}\rangle = \frac{1}{\sqrt{q}} \sum_{c=0}^{q-1} e^{i(\frac{2\pi}{q} + \delta_c)a} (1 + \delta'_c) |\tilde{c}\rangle, \left(\frac{1}{q^2} \sum_m \sum_k (1 + \delta_m)(1 + \delta_k) \cos\left[\frac{2\pi}{q} r(m-k) + (mr+l)\delta'_m - \right. \right.$$

where δ_c and δ'_c denote the error of A_j and B_{jk} , respectively.

Let us assume the following error modes: 1) systematic errors, where δ_c or δ'_c in (1) can only have systematic errors (EM₁); 2) random errors (EM₂), for which we assume that δ_c or δ'_c can only be random errors of the Gaussian or the uniform type; 3) coexistence of both systematic and random errors (EM₃). In the next section, we shall present the results of numerical simulations and discuss the effects of imperfect gate operation on the DFT algorithm, and thus on the Shor's algorithm.

III. INFLUENCE OF IMPERFECT GATE OPERATIONS

We first discuss the influence of imperfect gate operations in the initial preparation

$$\begin{aligned} A_{l-1}A_{l-2}\dots A_0|0\dots 0\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle + \delta_1(|0\rangle - |1\rangle)) \otimes (|0\rangle + |1\rangle + \delta_2(|0\rangle - |1\rangle)) \dots \\ &\doteq \frac{1}{\sqrt{2^l}} \sum_{i_1 i_2 \dots i_n=0}^1 |i_1 i_2 \dots i_n\rangle + \frac{1}{\sqrt{2^l}} \sum_{R=1}^n \delta_n \sum \end{aligned}$$

If the errors are systematic, for instance, caused by the inaccurate calibration of the rotations, then $\delta_1 = \delta_2 = \dots = \delta_n = \delta$. In this case, we can write the 2nd term as

$$|\psi\rangle = \frac{1}{\sqrt{2^l}} \delta \sum_{i_1 i_2 \dots i_n=0}^1 (2s-n) |i_1 i_2 \dots i_n\rangle,$$

where s stands for the number of 1's, and $2s-n = s - (n-s)$ is the difference in the number of 1's and 0's. Thus the results after the first procedure is

$$\frac{1}{\sqrt{2^l}} \sum_{a=0}^{2^L-1} (|a\rangle + \delta(2s-n)|a\rangle) = \frac{1}{\sqrt{2^l}} \sum_{a=0}^1 (1 + \delta_a)|a\rangle. \quad (2)$$

This implies that after the procedure, the amplitude of each state is no longer equal, but have slight difference. Combining the effect in the initialization and in the DFT, we have

$$(1 + \delta_a)(1 + \delta_c) e^{i(\frac{2\pi}{q} + \delta'_c)a} \doteq (1 + \delta'') e^{i(\frac{2\pi}{q} + \delta'_c)a},$$

where $\delta'' = \delta_c + \delta_a$. In the DFT, we have

$$|\psi\rangle \Rightarrow \frac{\sqrt{r}}{q} \sum_{c=0}^{q-1} \sum_{j=0}^{\frac{q}{r}-1} (1 + \delta_j) e^{i(\frac{2\pi}{q} + \delta'_j)(jr+l)} |\tilde{c}\rangle,$$

where we have rewrite δ'' as δ_j here. Let P_c denote the probability of getting the state $|\tilde{c}\rangle$ after we perform a measurement, we have

$$P_c = \frac{r}{q^2} \sum_{m=0}^{\frac{q}{r}-1} \sum_{k=0}^{\frac{q}{r}-1} (1 + \delta_m)(1 + \delta_k) e^{i(\frac{2\pi}{q} + \delta'_m)(mr+l)} \times e^{-i(\frac{2\pi}{q} + \delta'_k)(kr+l)} \left(\frac{1}{q^2} \sum_m \sum_k (1 + \delta_m)(1 + \delta_k) \cos\left[\frac{2\pi}{q} r(m-k) + (mr+l)\delta'_m - \right. \right.$$

From Eq. (3), we find that after the last measurement, each state can be extracted with a probability which is nonzero, and the offset l can't be eliminated.

Eq. (3) is very complicated, so we will make some predigestions to discuss different error modes for convenience. Generally speaking, the influence of exponential error δ_j is more remarkable than δ_j , so we can omit the error δ_j , thus

$$\text{DFT}_q |\phi\rangle = \frac{\sqrt{r}}{q} \sum_{c=0}^{q-1} \sum_{j=0}^{\frac{q}{r}-1} e^{i(\frac{2\pi}{q} + \delta'_j)(jr+l)} |c\rangle.$$

A. Case 1

If only systematic errors (EM₁) are considered, namely, all the δ_j 's are equal, then $\tilde{f}(c)$ can be given analytically

$$\begin{aligned}\tilde{f}(c) &= \frac{\sqrt{r}}{q} \sum_{j=0}^{\frac{q}{r}-1} e^{i(\frac{2\pi c}{q} + \delta)(jr+l)} \\ &= \frac{\sqrt{r}}{q} e^{il(\frac{2\pi c}{q} + \delta)} \frac{1 - e^{i(\frac{2\pi c}{q} + \delta)q}}{1 - e^{i(\frac{2\pi c}{q} + \delta)r}}\end{aligned}\quad (4)$$

The relative probability of finding c is

$$P_c = \left| \tilde{f}(c) \right|^2 = \frac{r}{q^2} \frac{\sin^2(\frac{\delta q}{2})}{\sin^2(\frac{\pi c r}{q} + \frac{\delta r}{2})},$$

and if $c = k\frac{q}{r}$, then

$$P_c = \frac{r \sin^2(\frac{\delta q}{2})}{q^2 \sin^2(\frac{\delta r}{2})}.$$

It can be easily seen that $\lim_{\delta \rightarrow 0} P_c = \frac{1}{r}$, which is just the case that no error is considered.

When δ takes certain values, say, $\delta = \frac{2}{r}(k - \frac{r}{q})\pi$ where k is an integer, then the summation in Eq. (4) is on longer valid. In our simulation, δ does not take these values. Here we consider the case where $q = 2^7 = 128$ and $r = 4$. For comparisons, we have drawn the relative probability for obtaining state c in Fig.1. for this given example. We have found the following results:

(i) When δ is small, the errors do hardly influence the final result, for instance when $c = k\frac{q}{r}$, then

$$\lim_{\delta \rightarrow 0} P_c = \lim_{\delta \rightarrow 0} \frac{r \sin^2(\frac{\delta q}{2})}{q^2 \sin^2(\frac{\delta r}{2})} = \frac{1}{r}.$$

The probability distribution is almost identical to those without errors.

(ii) Let us increase δ gradually, from Fig.2, we see that a gradual change in the probability distribution takes place. (Here, we again consider the relative probabilities) When δ is increased to certain values, the positions of peaks change greatly. For instance at $\delta = 0.05$, there appears a peak at $c=127$, whereas it is $P_c = 0$ when no systematic errors are present. In general, the influence of systematic errors on the algorithm is a shift of the peak positions. This influences the final results directly.

B. Case 2

When both random errors and systematic errors are present, we add random errors to the simulation. To see

the effect of different mode of random errors, we use two random number generators. One is the Gaussian mode and the other is the uniform mode. In this case, the error has the form $\delta = \delta_0 + s$, where δ_0 is the systematic error. s has a probability distribution with respect to c , depending on the uniform or the Gaussian distribution. When $\delta_0 = 0$, we have only random errors which is our error mode 2. When $\delta_0 \neq 0$, we have error mode 3. For the uniform distribution, $s \sim \pm s_{max} \times u(0, 1)$ where $u(0, 1)$ is evenly distributed in $[0, 1]$. s_{max} indicates the maximum deviation from δ_0 . For Gaussian distribution, $s \sim N(0, \sigma_0)$. Through the figure, we see the following:

(1) When only random errors are present ($\delta_0 = 0$), the peak positions are not affected by these random errors. However, different random error modes cause similar results. The results for uniform random error mode are shown in Fig.3. For the uniform distribution error mode, with increasing δ_{max} , the final probability distribution of the final results become irregular. In particular, when δ_{max} is very large, all the patterns are destroyed and is hardly recognizable. Many unexpected small peaks appear. For the Gaussian distribution error mode, as shown in Fig.4, the influence of the error is more serious. This is because in Gaussian distribution, there is no cut-off of errors. Large errors can occur although their probability is small. The influence of σ_0 on the final results is also sensitive, because it determines the shape of the distribution. When σ_0 increases, the final probability distribution becomes very messy. A small change in σ_0 can cause a big change in the final results.

(2) When $\delta_0 \neq 0$, which corresponds to error mode 3, the effect is seen as to shift the positions of the peaks in addition to the influences of the random errors.

IV. SUMMARY

To summarize, we have analyzed the errors in Shor's factorization algorithm. It has been seen that the effect of the systematic errors is to shift the positions of the peaks, whereas the random errors change the shape of the probability distribution. For systematic errors, the shape of the distribution of the final results is hardly destroyed, though displaced. We can still use the result with several trial guesses to obtain the right results because the peak positions are shifted only slightly. However, the random errors are detrimental to the algorithm and should be reduced as much as possible. It is different from the case with Grover's algorithm where systematic errors are disastrous while random errors are less harmful [10].

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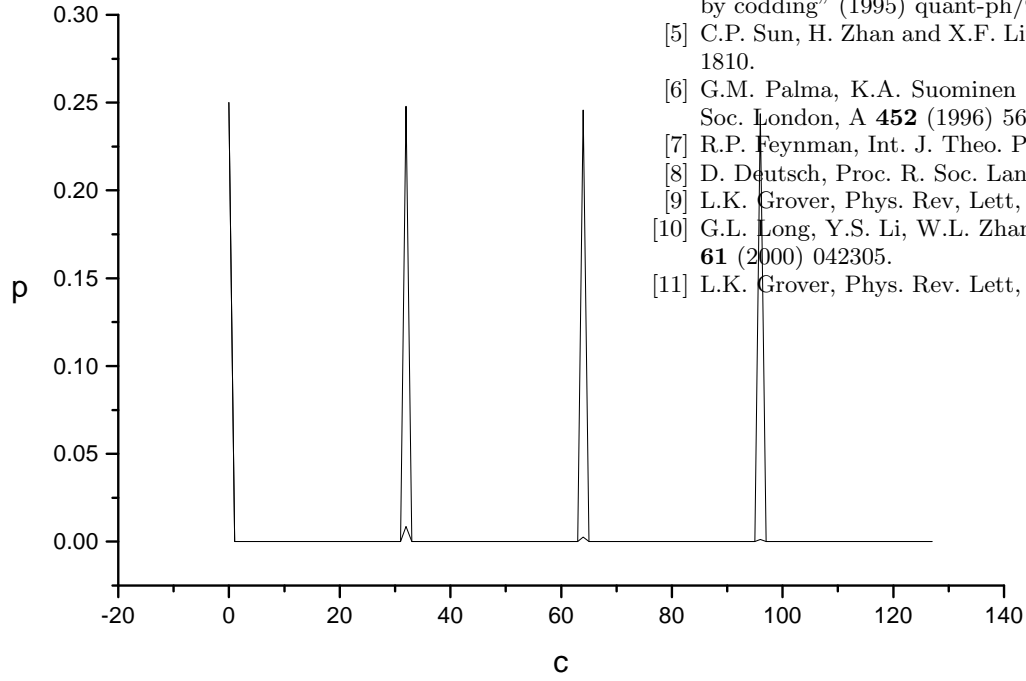


FIG. 1: Relative probability for finding state c in the absence of errors.

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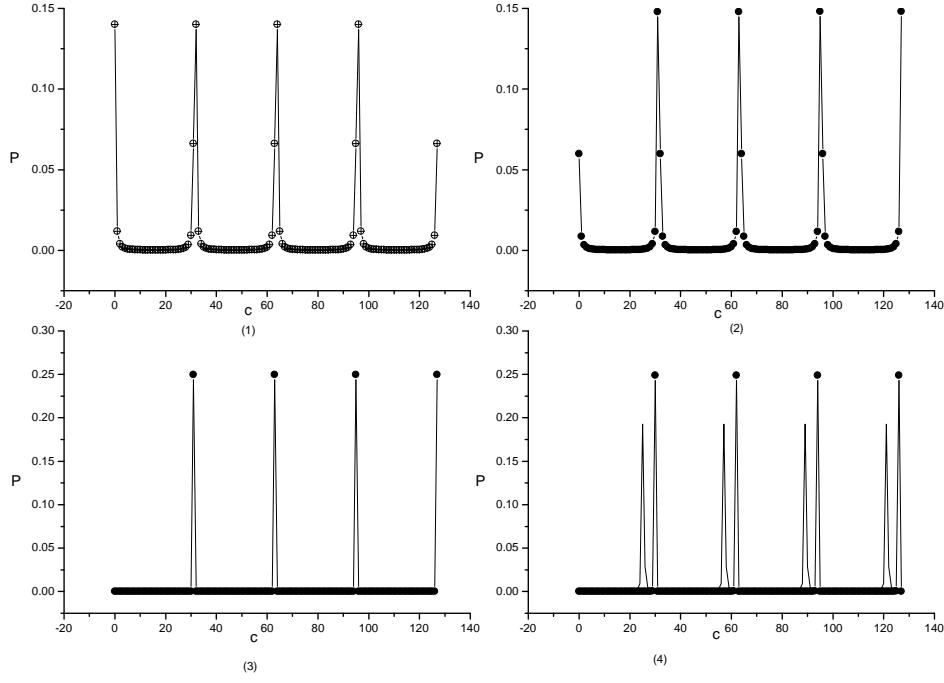


FIG. 2: The same as Fig.1. with systematic errors. In sub-figures (1), (2), (3), (4), δ are 0.02, 0.03, 0.05 respectively. In sub-figure (4), the curve with solid circles(with higher peaks) is the result with $\delta = 0.1$, and the one without solid circles(with lower peaks) denotes the result with $\delta = 0.33$.

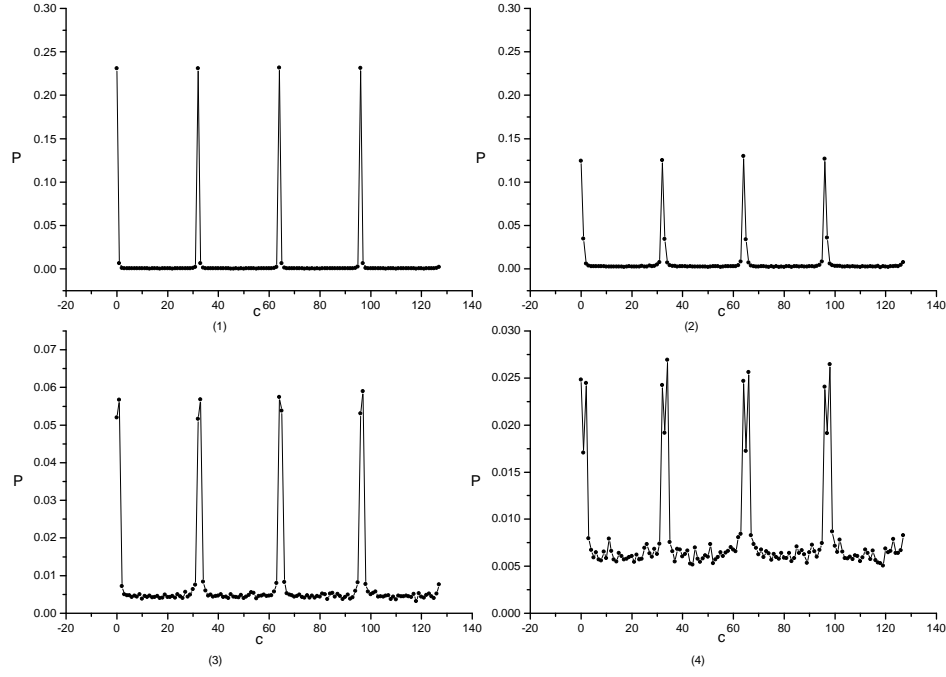


FIG. 3: The same as Fig.1. with uniform random errors. In sub-figures (1), (2), (3), (4), s_{max} are set to 0.01, 0.03, 0.05, 0.1 respectively.

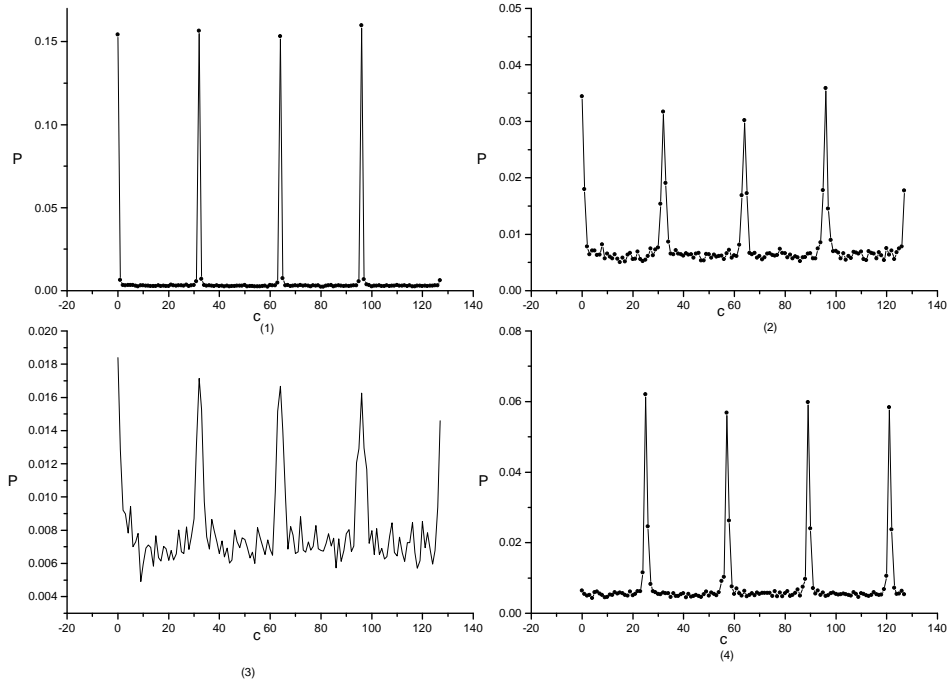


FIG. 4: The same as Fig.1. with Gaussian random errors and systematic errors. In sub-figures (1), (2), and (3) τ are set to 0.01, 0.03 and 0.05 respectively, and $\delta_0 = 0$ (without systematic errors). In sub-figure (4), both systematic and random Gaussian errors exist, where $\delta_0 = 0.33$, $\tau = 0.02$.